Office of Naval Research

Contract N5ORI-76 • Task Order No.1 • NR-078-011

VARIATIONAL APPROXIMATIONS TO THE DIFFRACTION BY CIRCULAR AND ELLIPTICAL APERTURES

By

Chaang Huang

June 5, 1953

Technical Report No. 164

Cruft Laboratory
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The research reported in this document was made possible through support extended Cruft Laboratory, Harvard University, jointly by the Navy Department (Office of Naval Research), the Signal Corps of the U. S. Army, and the U. S. Air Force, under ONR Contract N5ori-76, T. O. 1 and 28.

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TR164

Variational Approximations to the Diffraction by Circular and Elliptical Apertures

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Abstract

Different methods are reviewed briefly for attacking the problem of the diffraction of a plane electromagnetic wave by a circular aperture in a plane screen which is infinitesimally thin and perfectly conducting. Specifically, the variational method derived by Levine and Schwinger has been used to evaluate the transmission coefficient of circular and elliptical apertures. For the circular aperture, a high-order vector trial function with undetermined frequency-dependent coefficients is chosen. By using the stationary property of the expression for the transmission coefficient, equations for the undetermined coefficients are derived. These equations are solved to give a first-order approximation of the transmission coefficient, and the numerical values are compared with previous results. For elliptical apertures, a zeroth-order approximation of the transmission coefficient is evaluated using a one-component trial function. Numerical results are given for ellipses with minor-to-major-axis ratios of 1/2 and 1/3.

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I.

Introduction

The problem of the diffraction of electromagnetic waves by a circular aperture in an infinite plane conducting screen has attracted the attention of many authors. Their work is reviewed briefly in this section.

Moglich[1] tried to work out the exact solution for the problem of diffraction by a circular disk (which is the Babinet[2] complement of a circular aperture) by expanding the rectangular components of the Hertz vector, π, of the incident and scattered fields in series of oblate spheroidal wave func-

tions and then imposing the boundary condition on π at the surface of the disk to determine the unknown coefficients of the expansion for the scattered Hertz vector. His solution was incorrect because he did not take into consideration the singularity at the rim of the disk. This defect was pointed out by Meixner,[3] who reconsidered the problem by expanding the Debye potentials into series of oblate spheroidal wave functions, and later improved his own result with the collaboration of Andrejewski[4] by employing the Hertz vectors. Their solution includes the complementary problem, in the Babinet sense, of diffraction by a circular aperture. The analogous exact solution to the more general problem of the elliptical disk awaits some future study of ellipsoidal wave functions.[1]

Although exact solutions for an arbitrarily shaped disk or aperture are not feasible at present, a number of different formulations are available for obtaining approximate results over specific frequency ranges. The application of these methods to the problem of a circular disk or aperture permits a comparison with the results of the exact theory and gives some idea of the degree of accuracy to be expected from a particular approximation. It is to be emphasized, however, that the significance of these methods lies in their applicability to disks or apertures that are not circular in shape.

To the extent that they are solutions of Maxwell's equations all these formulations of the problem of diffraction are equivalent. When explicit results are desired for a particular case, however, the necessary approximations give differing degrees of accuracy, depending upon the frequency range that is of interest and the formulation that is used. Approximations which have proved useful in the past can be classified as follows:

1. The low-frequency (static) approximation: The aperture or disk is taken to be small compared to the wavelength; $ka = 2\pi a/\lambda << 1$ (where a = characteristic dimension of the aperture or disk).

a. Retardation is neglected, and the result gives the first few terms in an expansion in powers of ka.

b. The incident field vectors are taken to be constant over the small aperture or on the surface of a small disk.

Using these approximations Rayleigh,[5] Tai,[6] Bethe[*] [7] and Bouwkamp[8] have studied the problem of diffraction by small circular disks or apertures. The results obtained by these investigators are accurate only for frequencies which satisfy the condition $ka \ll 1$.

2. The high-frequency (optical) approximation: The aperture or disk is taken to be large compared to the wavelength; $ka \gg 1$. These approximations are of the Kirchhoff type.

a. The tangential component of the magnetic field vanishes on the dark side of the conducting screen.

b. The tangential component of the electric field in the aperture is equal to the incident electric field.

Formulations which make use of these approximations have been discussed by Stratton and Chu,[10] Schelkunoff,[11] Silver and Ehrilich,[**] [12] and Levine and Schwinger.[13] Among these four, the analysis of Levine and Schwinger has the advantage of a relative analytic simplicity, and is recapitulated in a previous report,[14] where its application to annular, elliptical and rectangular apertures is worked out.

In contrast to the above methods, Levine and Schwinger[13] have derived vector integral equations[***] using dyadic Green's functions, and they have shown how to calculate the far-zone diffracted fields and the transmission coefficients of apertures in terms of variational principles related to the integral equations. Thus, by choosing an appropriate trial function for either the tangential component of the electric field in the aperture or for the magnetic field on the back side of the diffracting screen, an accurate result can be obtained for the transmission coefficient without making any further assumptions. In this way, they have computed the transmission coefficient of a circular aperture for normal incidence, using two different trial functions for the tangential component of the electric field in the aperture.

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* Bethe's solution has been modified by Bouwkamp.[9]

**Silver and Ehrlich have solved the Maxwell's equation by a Fourier operational method and obtained formulas for the near-zone fields.

***Copson[15] has also derived the integral equations for the diffraction of disks and apertures, but since he uses a scalar Green's function, his formulation is very cumbersome.

The first of these functions was constructed from a single-component electric field with no angular variation and with a radial dependence corresponding to that of the static solution. It results in a transmission coefficient that is accurate only at low frequencies. The second trial function is the complete static aperture field obtained by Bouwkamp. It gives an improved result which is still accurate at moderately high frequencies where the Bouwkamp solution itself is not valid.

Since these trial functions are frequency-independent, they give unsatisfactory answers in the middle frequencies. In this study, a frequency-dependent trial function will be constructed for the circular aperture, and it will be seen that a very high degree of accuracy can be obtained with only moderate complexity. Finally, the variational procedure will be applied to elliptical apertures using a single-component trial function analagous to the first one used by Levine and Schwinger for the circular aperture.

II.

The Circular Aperture

The Stationary Expression for the Transmission Coefficient

The problem to be considered is that of a plane electromagnetic wave incident on an aperture S_1 which perforates an infinitesimally thin, perfectly conducting plane screen, S_2. A rectangular coordinate system (x, y, z) is chosen so that S_1 and S_2 lie in the plane $z = 0$ as shown in Fig. 1.

The plane wave is incident in the half-space $z<0$ and is described by

$$\begin{aligned} E^{inc}(r) &= \hat{e}\exp(ikn\cdot r), \\ H^{inc}(r) &= \hat{h}\exp(ikn\cdot r), \end{aligned} \tag{1}$$

where r is a position vector in space, $\hat{n}$ is the unit propagation vector, $\hat{e}$ and $\hat{h}$ are unit polarization vectors, and k is the wave number. In gaussian units the unit vectors are related by

$$\hat{e} = \hat{h}\times\hat{n}, \quad \text{and} \quad \hat{e}\cdot\hat{e} = \hat{h}\cdot\hat{h} = 1$$

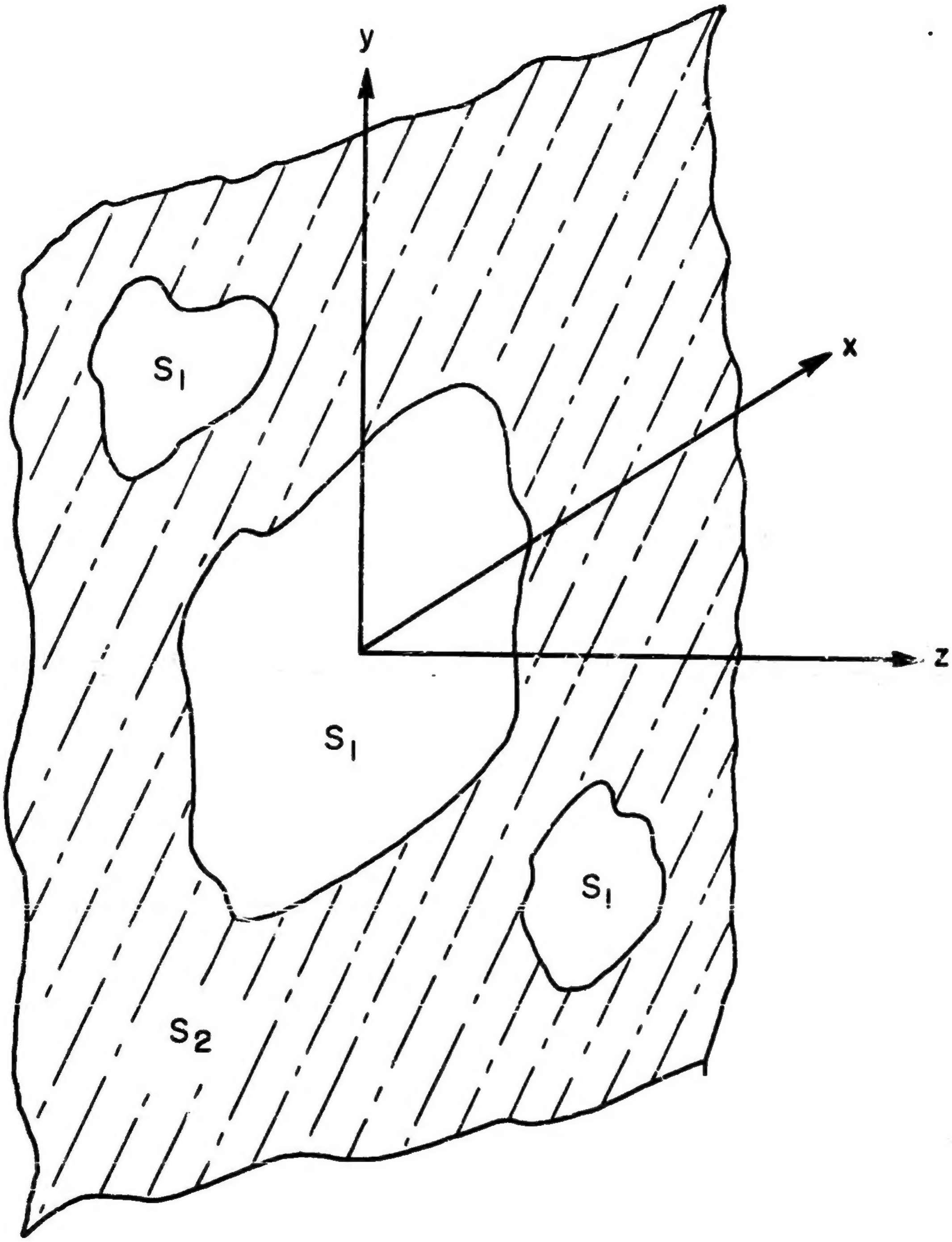

FIG. 1 DIFFRACTING APERTURES IN A PLANE SCREEN

The harmonic time dependence $e^{-i\omega t}$ is omitted here and will be omitted throughout this report.

The transmission coefficient of an aperture is defined as the ratio of the transmitted energy flux per unit aperture area to incident energy flux per unit area. A stationary expression for this quantity has been derived by Levine and Schwinger;[13] it is

$$t = \frac{-1}{2kA}\,\mathrm{Im}\left\{\frac{\left(\hat{h}\cdot\int_{S_1}\hat{z}\times E_n(\rho)\exp(-ik\hat{n}\cdot\rho)\,dS\right)\left(\hat{h}\cdot\int_{S_1}\hat{z}\times E_{-n}(\rho)\exp(ik\hat{n}\cdot\rho)\,dS\right)}{\int_{S_1}\hat{z}\times E_n(\rho)\cdot\Gamma^{(0)}(\rho,\rho')\cdot\hat{z}\times E_{-n}(\rho)\,dS\,dS'}\right\} \tag{2}$$

where ρ is the position vector in the aperture and A is the area of the aperture. $E_n(\rho)$ and $E_{-n}(\rho)$ are the aperture electric fields produced by incident plane waves with propagation vectors $\hat{n}$ and $-\hat{n}$ respectively, both having the magnetic polarization vector $\hat{h}$. $\Gamma^{(0)}(\rho,\rho')$ is the free-space dyadic Green's function and is given by

$$\Gamma^{(0)}(\rho,\rho') = \left(\epsilon - \frac{1}{k^2}\nabla\nabla'\right)\frac{\exp(ik|\rho-\rho'|)}{4\pi|\rho-\rho'|}\ , \tag{3}$$

where ϵ = unit dyadic = $\hat{x}\hat{x}' + \hat{y}\hat{y}'$.

If a plane wave is normally incident on a circular aperture of radius $\underline{a}$, then it is evident that $A = \pi a^2$, $\hat{n} = \hat{z}$, and $\hat{n}\cdot\rho = 0$. The symmetry of the circular aperture permits an arbitrary polarization for the incident wave, and it is convenient to choose $\hat{h} = \hat{y}$.

Furthermore, in polar coordinates the aperture will be described by

$$S_1 : \qquad 0 \leqslant \rho \leqslant a \qquad\qquad 0 \leqslant \phi \leqslant 2\pi$$

With these specializations, the stationary expression for the transmission coefficient (2) reduces to

$$t = \frac{-1}{2k\pi a^2}\,\mathrm{Im}\left\{\frac{\left(\hat{y}\cdot\int_0^a\int_0^{2\pi}\rho\,d\rho\,d\phi\,\hat{z}\times E_z(\rho)\right)\left(\hat{y}\cdot\int_0^a\int_0^{2\pi}\rho\,d\rho\,d\phi\,\hat{z}\times E_{-z}(\rho)\right)}{\int_0^a\int_0^{2\pi}\rho\,d\rho\,d\phi\,\rho'\,d\rho'\,d\phi'\,\hat{z}\times E_z(\rho)\cdot\Gamma^{(0)}(\rho,\rho')\cdot z\times E_{-z}(\rho')}\right\} \tag{4}$$

The Trial Function

Before choosing appropriate trial functions for the electric fields in the aperture, it will be worth while to examine some general properties of the aperture fields. The fields in question are produced by a pair of plane waves propagating in the directions $\hat{z}$ and $-\hat{z}$ with magnetic polarization vectors in the same direction parallel to the y-axis. The tangential component of the aperture field excited by the $\hat{z}$-directed wave can be assumed to be separable in polar coordinates, and to be given by

$$\hat{z} \times E_z(\rho,\phi) = \hat{z} \times [\hat{\rho} R_\rho(\rho)\Phi_\rho(\phi) + \hat{\phi} R_\phi(\rho)\Phi_\phi(\phi)] \,. \tag{5}$$

Then, as a consequence of the symmetry of the incident waves and of the aperture, sketched in Fig. 2, it can be seen that the two tangential components of the aperture field associated with the $\hat{z}$-directed wave are

$$\begin{aligned} \hat{\rho}\cdot E_{-z}(\rho,\pi-\phi) &= \hat{\rho}\cdot E_z(\rho,\phi) = R_\rho(\rho)\,\Phi_\rho(\phi) \\ \hat{\phi}\cdot E_{-z}(\rho,\pi-\phi) &= -\hat{\phi}\cdot E_z(\rho,\phi) = -R_\phi(\rho)\,\Phi_\phi(\phi) \,. \end{aligned} \tag{6}$$

The second field can, therefore, be expressed in terms of the components of the first, i.e.,

$$\hat{z} \times E_{-z}(\rho,\pi-\phi) = \hat{z} \times [\hat{\rho}\, R_\rho(\rho)\,\Phi_\rho(\phi) - \hat{\phi} R_\phi(\rho)\,\Phi_\phi(\phi)] \,. \tag{7}$$

If it is assumed that the trial fields have the same ϕ-dependence as the incident field, then

$$\Phi_\rho(\phi) = \cos\phi \;, \quad \Phi_\phi(\phi) = -\sin\phi \,. \tag{8}$$

With (5), (7) and (8) the aperture field functions at the point (ρ,ϕ) become

$$\begin{aligned} \hat{z} \times E_z(\rho,\phi) &= \hat{z} \times [\,\hat{\rho}\, R_\rho(\rho)\cos\phi - \hat{\phi}\, R_\phi(\rho)\sin\phi] \\ \hat{z} \times E_{-z}(\rho,\phi) &= \hat{z} \times [\,\hat{\rho}\, R_\rho(\rho)\cos(\pi-\phi) + \hat{\phi}\, R_\phi(\rho)\sin(\pi-\phi)] \,, \end{aligned} \tag{9}$$

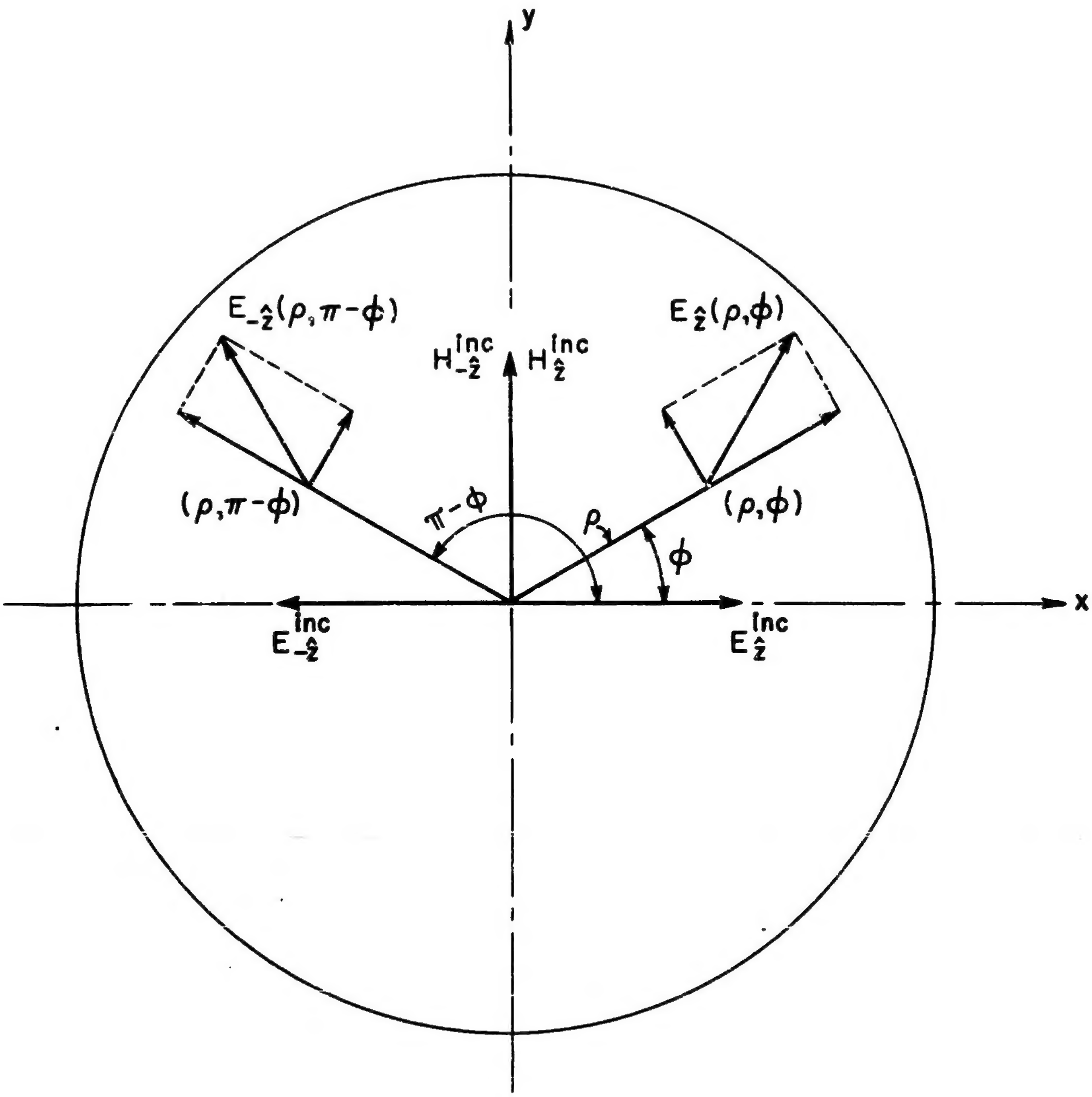

FIG. 2 THE TANGENTIAL APERTURE ELECTRIC FIELDS OF OPPOSITE INCIDENCE WITH PARALLEL MAGNETIC POLARIZATION VICTORS

so that

$$\hat{z} \times E_z(\rho,\phi) = -\hat{z} \times E_{-z}(\rho,\phi) = \hat{z} \times E(\rho,\phi) . \tag{10}$$

A reasonable choice for the ρ-dependence of the trial field is one which satisfies the same boundary conditions as those satisfied by the correct field. These conditions have been fully discussed by Meixner[16] and Bouwkamp[17] who show that at the rim of an aperture the tangential component of the electric field vanishes as $R^{1/2}$ and the normal component increases as $R^{-1/2}$, where R measures the distance from the field point to the edge. Accordingly, possible series representations of the radial functions associated with the aperture field components are

$$\begin{aligned} R_\rho(\rho) &= \frac{1}{\sqrt{1-(\frac{\rho}{a})^2}} \sum_{n=0}^{\infty} a_n (\tfrac{\rho}{a})^{2n} \\ R_\phi(\rho) &= \sqrt{1-(\tfrac{\rho}{a})^2} \sum_{n=0}^{\infty} b_n (\tfrac{\rho}{a})^{2n} , \end{aligned} \tag{11}$$

where a_n, b_n are undetermined coefficients.

With (8) and (11), the ρ- and ϕ-components of the trial functions for the aperture field are :

$$\begin{aligned} \hat{\rho} \cdot E(\rho,\phi) &= \frac{1}{\sqrt{1-(\frac{\rho}{a})^2}} \sum_{n=0}^{\infty} a_n (\tfrac{\rho}{a})^{2n} \cos\phi \\ \hat{\phi} \cdot E(\rho,\phi) &= \sqrt{1-(\tfrac{\rho}{a})^2} \sum_{n=0}^{\infty} b_n (\tfrac{\rho}{a})^{2n} \sin\phi \end{aligned} \tag{12}$$

The corresponding rectangular components are :

$$\hat{x} \cdot E(\rho,\phi) = \frac{1}{\sqrt{1-(\frac{\rho}{a})^2}} \sum_{n=0}^{\infty} a_n (\tfrac{\rho}{a})^{2n} \cos^2\phi - \sqrt{1-(\tfrac{\rho}{a})^2} \sum_{n=0}^{\infty} b_n (\tfrac{\rho}{a})^{2n} \sin^2\phi$$

$$\hat{y}\cdot E(\rho,\phi) = \left[\frac{1}{\sqrt{1-(\frac{\rho}{a})^2}}\sum_{n=0}^{\infty} a_n(\frac{\rho}{a})^{2n} + \sqrt{1-(\frac{\rho}{a})^2}\sum_{n=0}^{\infty} b_n(\frac{\rho}{a})^{2n}\right]\sin\phi\cos\phi \tag{13}$$

At the center of the aperture these components reduce to

$$\hat{x}\cdot E(\rho,\phi) = \cos^2\phi\,(a_o + b_o) - b_o$$

$$\hat{y}\cdot E(\rho,\phi) = (a_o + b_o)\sin\phi\cos\phi \tag{14}$$

But since ϕ is not defined at the origin, the rectangular components should be independent of ϕ when $\rho = 0$. It is necessary, therefore, to require that

$$a_o = -b_o \tag{15}$$

This condition also insures that $\nabla\cdot E = 0$ at the center of the aperture.

If the series are truncated so that $a_n = 0$ for $n \geqslant 2$, $b_n = 0$ for $n \geqslant 1$, and the remaining coefficients are assigned the values $a_1 = -a$, $a_o = -b_o = 2a$, the trial function reduces to

$$\hat{\rho}\cdot E(\rho,\phi) = \frac{2a^2 - \rho^2}{\sqrt{a^2 - \rho^2}}\cos\phi$$

$$\hat{\phi}\cdot E(\rho,\phi) = -2\sqrt{a^2 - \rho^2}\sin\phi \tag{16}$$

These functions constitute the low-frequency exact solution obtained by Bouwkamp.[8] They are frequency independent and are valid only for $ka \ll 1$. In reference (13), these functions are used as a zeroth-order trial function by Levine and Schwinger.

Determination of the Coefficients in the Vector Trial Function

The vector trial function $\hat{z} \times E(\rho)$ that was chosen in the last section contains undetermined frequency-dependent coefficients. These may be chosen appropriately by using the stationary property of the expression for the transmission coefficient.

If the transmission coefficient of (4) is written

$$t = \frac{2}{ka^2} \ \mathrm{Im}\, A_y = -\frac{2}{ka^2} \ \mathrm{Im}\, \frac{I_1^2}{4\pi I_2} , \tag{17}$$

then the substitution of the explicit series for $\hat{z} \times E(\rho)$ shows at once that

$$I_1 = \int_0^a \int_0^{2\pi} \rho\, d\rho\, d\phi\, \hat{y} \cdot \hat{z} \times E(\rho) = \sum_{n=0}^{\infty} (a_n B_n^a + b_n B_n^b)$$

$$I_2 = \int_0^a \int_0^{2\pi} \rho\, d\rho\, d\phi\, \rho'\, d\rho'\, d\phi'\ \hat{z} \times E(\rho) \cdot \Gamma^{(0)}(\rho, \phi; \rho', \phi') \cdot \hat{z} \times E(\rho')$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [a_n a_m C_{nm}^{aa} + a_n b_m C_{nm}^{ab} + b_n b_m C_{nm}^{bb}]$$

where

$$B_n^a = \pi \int_0^a \frac{\rho\, d\rho}{\sqrt{1 - (\frac{\rho}{a})^2}} \ (\frac{\rho}{a})^{2n}$$

$$B_n^b = -\pi \int_0^a \rho \sqrt{1 - (\frac{\rho}{a})^2}\ (\frac{\rho}{a})^{2n}\, d\rho$$

$$C_{nm}^{aa} = \int_{S_1} dS\, dS' \frac{(\frac{\rho}{a})^{2n} (\frac{\rho'}{a})^{2n}}{\sqrt{1 - (\frac{\rho}{a})^2}\sqrt{1 - (\frac{\rho'}{a})^2}} \ \cos\phi \cos\phi'\ \Gamma_{\phi\phi}^{(0)}(\rho, \phi; \rho', \phi') = C_{mn}^{aa}$$

$$C_{nm}^{ab} = -\int_{S_1} dS\, dS' \frac{\sqrt{1 - (\frac{\rho}{a})^2}}{\sqrt{1 - (\frac{\rho'}{a})^2}} \ (\frac{\rho}{a})^{2m} (\frac{\rho'}{a})^{2n} \sin\phi \cos\phi'\ \Gamma_{\rho\phi}^{(0)}(\rho, \phi; \rho', \phi')$$

$$-\int_{S_1} dS\, dS' \frac{\sqrt{1 - (\frac{\rho'}{a})^2}}{\sqrt{1 - (\frac{\rho}{a})^2}} \ (\frac{\rho}{a})^{2n} (\frac{\rho'}{a})^{2m} \cos\phi \sin\phi'\ \Gamma_{\phi\rho'}^{(0)}(\rho, \phi; \rho', \phi') = C_{mn}^{ab}$$

$$C_{nm}^{bb} = \int_{S_1} dS\, dS' \sqrt{1-(\frac{\rho}{a})^2}\sqrt{1-(\frac{\rho'}{a})^2}\,(\frac{\rho}{a})^{2n}(\frac{\rho}{a})^{2m}\sin\phi\sin\phi\,\Gamma^{(0)}_{\rho\rho'}(\rho\phi;\rho',\phi) = C_{mn}^{bb}$$

The $\Gamma^{(0)}_{uv}$ are the $\hat{u}\hat{v}$ components of the free-space dyadic Green's function. With these substitutions the scattered amplitude A_y takes the form:

$$A_y = \frac{-1}{4\pi}\,\frac{\sum_{n=0}^{\infty}(a_nB_n^a + b_nB_n^b)^2}{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[a_na_mC_{nm}^{aa} + a_nb_mC_{nm}^{ab} + b_nb_mC_{nm}^{bb}]},$$

where the B's and C's are, in principle, known functions of the wavelength and the size of the aperture, while the a's and b's remain to be determined.

The procedure is to differentiate A_y with respect to each of the independent coefficients and set

$$\frac{\partial A_y}{\partial a_\nu} = \frac{\partial A_y}{\partial b_\mu} = 0$$

Since the behavior of the trial function at the center of the aperture requires $a_o = -b_o$ extremizing A_y with respect to either of these coefficients gives

$$A_y \sum_{m=0}^{\infty}[2a_mC_{om}^{aa} - (a_m - b_m)\,C_{om} - 2b_mC_{om}^{bb}]$$

$$= \frac{-1}{2\pi}(B_o^a - B_o^b)\sum_{m=0}^{\infty}[a_mB_m^a + b_mB_m^b] \qquad (19)$$

But by definition

$$A_y = \frac{ik}{2\pi} I_1 = \frac{ik}{2\pi} \sum_{m=0}^{\infty} (a_m B_m^a + b_m B_m^b) \tag{20}$$

so that

$$\sum_{m=0}^{\infty} [(2C_{om}^{aa} - C_{om}^{ab}) a_m - (2C_{om}^{bb} - C_{om}^{ab}) b_m] = \frac{i}{k} (B_o^a - B_o^b) . \tag{21}$$

A similar treatment with respect to a_ν and b_μ gives

$$\sum_{m=0}^{\infty} [2C_{\nu m}^{aa} a_m + C_{\nu m}^{ab} b_m] = \frac{i}{k} B_\nu^a \qquad \nu = 1, 2, \ldots \tag{22}$$

$$\sum_{m=0}^{\infty} [C_{\mu m}^{ab} a_m + 2C_{\mu m}^{bb} b_m] = \frac{i}{k} B_\mu^b \qquad \mu = 1, 2, \ldots \tag{23}$$

Equations (21), (22), and (23) constitute the linear set of algebraic equations required to determine the a_m and b_m. With these, the transmission coefficient can be calculated through (20).

In a practical calculation the trial function must be terminated at some convenient value of m. The approximate transmission coefficient obtained in this way will be

$$t^{(N)} = -\frac{1}{\pi a^2} \mathrm{Re} [\sum_{m=0}^{j} a_m B_m^a + \sum_{m=0}^{k} b_m B_m^b] , \tag{24}$$

where $N = j + k$ is the order of the approximation. For a first-order solution with $j = 1$, $k = 0$, the set of equations to be solved consists only of (21) and (22) with $\nu = 1$:

$$2 (C_{oo}^{aa} - C_{oo}^{ab} + C_{oo}^{bb}) a_o + (2C_{01}^{aa} - C_{01}^{ab}) a_1 = \frac{i}{k} (B_o^a - B_o^b) \tag{25}$$

$$(2C_{01}^{aa} - C_{01}^{ab}) a_o + 2C_{11}^{aa} a_1 = \frac{i}{k} B_1^a \tag{26}$$

These equations are readily solved for the coefficients a_0 and a_1, which can be substituted into (24). The result is

$$t^{(1)}(\alpha) = \frac{32\alpha}{9\pi} \, \mathrm{Im} \, \frac{4P(\alpha) - 2Q(\alpha) + R(\alpha)}{Q^2(\alpha) - 4P(\alpha)R(\alpha)} \qquad (27)$$

Here, $\alpha = ka$ and

$$P(\alpha) = C_{11}^{aa}$$

$$Q(\alpha) = 2C_{01}^{aa} - C_{01}^{ab}$$

$$R(\alpha) = C_{00}^{aa} - C_{00}^{ab} + C_{00}^{bb} .$$

These integrals are evaluated in Appendix A. In addition,

$$B_o^a = \pi a \int_o^a \frac{\rho \, d\rho}{\sqrt{a^2 - \rho^2}} = \pi a^2$$

$$B_1^a = \frac{\pi}{a} \int_o^a \frac{\rho^3 \, d\rho}{\sqrt{a^2 - \rho^2}} = \frac{2}{3} \pi a^2$$

$$B_o^b = - \frac{\pi}{a} \int_o^a \rho \sqrt{a^2 - \rho^2} \, d\rho = - \frac{1}{3} \pi a^2$$

The numerical result of $t^{(1)}$ is compared with previous results in Fig. 3. It can be seen that the variational approximations are quite close to the exact solution [18] and that $t^{(1)}$ provides a much higher degree of accuracy than $t^{(0)}$ in the range $1.5 \leqslant ka \leqslant 4.5$.

A further comparison of interest can be made for small values of ka. If the transmission coefficients obtained by the various theories are expanded into power series in ka , the results are :

(i) Exact Solution (due to Bouwkamp[8])

$$t_B = \frac{64(ka)^4}{27\pi^2} \left[1 + \frac{22}{25}(ka)^2 + 0.39793197\,(ka)^4 + \ldots \right] \qquad (28)$$

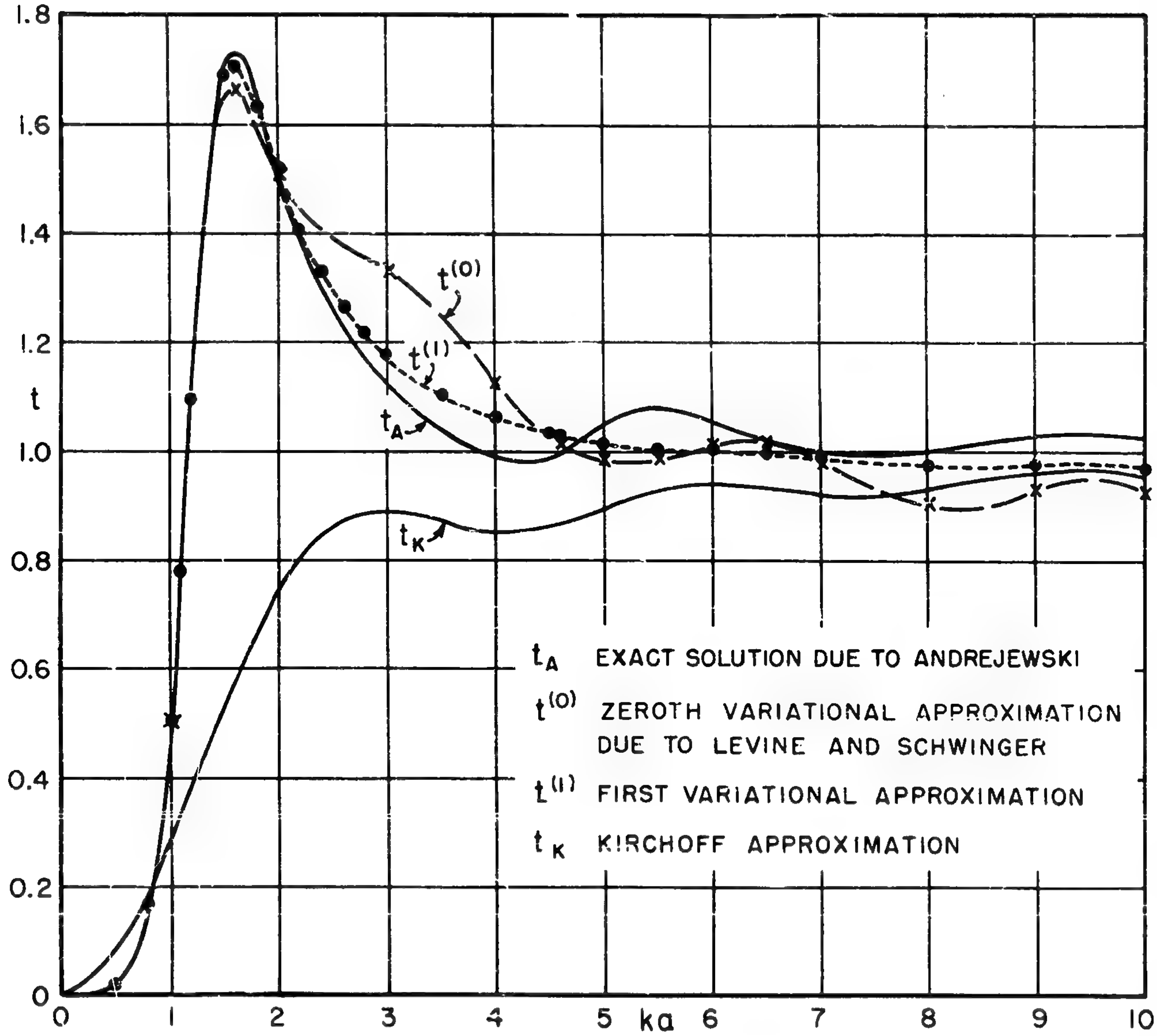

FIG. 3 TRANSMISSION COEFFICIENT OF CIRCULAR APERTURE FOR NORMAL INCIDENCE OF PLANE ELECTROMAGNETIC WAVES

(ii) Zeroth-order approximation by the variational method (due to Levine and Schwinger)[13]

$$t^{(0)} = \frac{64(ka)^4}{27\pi^2}\left[1 + \frac{22}{25}(ka)^2 + 0.40790023\,(ka)^4 + \ldots\right] \qquad (29)$$

(iii) First-order approximation by the variational method obtained in this report

$$t^{(1)} = \frac{64(ka)^4}{27\pi^2}\left[1 + \frac{22}{25}(ka)^2 + 0.39678912\,(ka)^4 + \ldots\right] \qquad (30)$$

The third terms in the bracket show that $t^{(1)}$ is more accurate than $t^{(0)}$.

III.

The Elliptical Aperture

If the aperture in the plane screen is elliptical in shape, the general procedure of section II is much more difficult to apply. Either elliptic or oblate spheroidal coordinates must be used, and the computational labor would be greatly increased. However, considerable improvement in the Kirchhoff result can be effected by using a single-component trial function for the electric field in the aperture corresponding to the original trial function of Levine and Schwinger in the circular case.

For a normally incident plane wave polarized so that $E^{inc}(r) = \hat{x}\, e^{ikz}$ and $H^{inc}(r) = \hat{y}\, e^{ikz}$, the choice of trial function is

$$E(\rho) = \hat{x}\,(1-\rho^2)^{\frac{1}{2}} \qquad (31)$$

Substitution of this function into the stationary expression for the transmission coefficient (4) gives

$$t = \frac{-2\pi a b}{9k}\, \mathrm{Im}\, \frac{1}{I} \qquad (32)$$

where

$$I = \int_0^1\!\!\int_0^{2\pi} \rho\, d\rho\, d\phi\, \rho'\, d\rho'\, d\phi' \sqrt{1-\rho^2}\,\sqrt{1-\rho'^2}\; \Gamma_{yy}^{(0)}(\rho,\phi;\rho',\phi') .$$

With the representation

$$\Gamma_{yy}^{(0)}(\rho,\phi;\rho',\phi') = \frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{dk_x\,dk_y}{\sqrt{k^2-k_x^2-k_y^2}}\left(1-\frac{k_y^2}{k^2}\right)\exp\left[ikx(x-x')+iky(y-y')\right],$$

the integral can be simplified to

$$I = \frac{a^2b^2\pi}{4k^2}\left\{\int_0^{2\pi}\int_0^{\infty}\frac{dv\,d\theta\sqrt{v^2-1}\;J_{3/2}^2\,(kLv)}{L^3v^2}\right.$$

$$\left.-\int_0^{2\pi}\int_0^{\infty}dv\,d\theta\;\frac{\cos^2\theta\;J_{3/2}^2\,(kLv)}{L^3\sqrt{v^2-1}}\right\} \qquad (34)$$

where

$$L = \sqrt{(a\cos\theta)^2+(b\sin\theta)^2}\;.$$

The procedure is similar to that developed in reference (14). Finally, by carrying out the integration with respect to v , we find that*

$$I = \frac{-a^2b^2\pi}{4k^2}\int_0^{2\pi}\left[F(kL)-\cos^2\theta\,G(kL)\right]\frac{d\theta}{L^3} \qquad (35)$$

where

$$F(x) = \frac{1}{4x}\left\{\left[\frac{-1}{2x}J_0(2x)-J_1(2x)+\left(1+\frac{1}{4x^2}\right)\int_0^{2x}J_0(t)\,dt\right]\right.$$

$$\left.-i\left[\frac{2x^2}{\pi}-\frac{1}{\pi}+\frac{1}{2x}S_0(2x)+S_1(2x)-\left(1+\frac{1}{4x^2}\right)\int_0^{2x}S_0(t)\,dt\right]\right\}$$

- - - - - - -

*The first integral is related to $F_{11}(kL)$ in reference (19); the second is related to $I_2(kL)$ in appendix A.

$$G(x) = \frac{1}{x}\left\{\left[\frac{-1}{2x} J_0(2x) - J_1(2x) + \left(\frac{1}{2} + \frac{1}{4x^2}\right) \int_0^{2x} J_0(t)\, dt\right]\right.$$

$$\left. + i\left[\frac{1}{\pi} - \frac{1}{2x} S_0(2x) - S_1(2x) + \left(\frac{1}{2} + \frac{1}{4x^2}\right) \int_0^{2x} S_0(t)\, dt\right]\right\}$$

Numerical values of the transmission coefficient are given in Fig. 4 for $0<ka\leq10$, and $b/a = 1/2, 1/3$. These are plotted together with the corresponding result for the circular aperture which has been given in reference (13), and which can be obtained from equation (32) by putting $b/a = 1$. Using the expansions

$$F(x) = \left[\frac{1}{3} - \frac{x^2}{15} + \frac{x^4}{140} + \ldots\right] - i\left[\frac{2x^3}{27\pi} - \frac{4x^5}{675\pi} + \frac{16x^7}{55125\pi} + \ldots\right]$$

$$G(x) = \left[\frac{1}{3} + \frac{x^2}{15} - \frac{3x^4}{140} + \ldots\right] + i\left[\frac{4x^3}{27\pi} - \frac{16x^5}{675\pi} + \frac{32x^7}{18375\pi} + \ldots\right] \tag{36}$$

in (35), a form of the transmission coefficient appropriate to small values of ka can be obtained :

$$t = \frac{64}{27\pi^2}\left(\frac{a}{b}\right)(ka)^4\left[A_0\left(\frac{b}{a}\right) + A_2\left(\frac{b}{a}\right)(ka)^2 + A_4\left(\frac{b}{a}\right)(ka)^4 + \ldots\right] \tag{37}$$

where $A_0(\frac{b}{a})$, $A_2(\frac{b}{a})$, $A_4(\frac{b}{a})$, ... are functions of the eccentricity. They are given in appendix B. With small eccentricity (37) becomes

$$t = \frac{64}{27\pi^2}\left(\frac{a}{b}\right)(ka)^4\left[\left(1 - \frac{9}{4}e^2 \ldots\right) + \left(\frac{27}{25} - \frac{349}{100}e^2 + \ldots\right)(ka)^2\right.$$

$$\left. + \left(\frac{8937}{12250} - \frac{155223}{49000}e^2 + \ldots\right)(ka)^4 + \ldots\right] \tag{38}$$

Appendix A

In this appendix we will sketch the method used to evaluate the functions required for the transmission coefficient of the circular aperture. These functions are given by (27):

$$P(a) = C_{11}^{aa}(a)$$

$$Q(a) = 2\,C_{01}^{aa}(a) - C_{01}^{ab}(a)$$

$$R(a) = C_{00}^{aa}(a) - C_{00}^{ab}(a) + C_{00}^{bb}(a)$$

The integrals C_{nm} can be simplified by making use of an integral representation and an addition theorem: the integral representation is[20]

$$\frac{e^{ik|\vec{\rho} - \vec{\rho}\,'|}}{4\pi|\vec{\rho} - \vec{\rho}\,'|} = \frac{1}{4\pi}\int_0^\infty J_o(\omega R)\,\frac{\omega\,d\omega}{\sqrt{\omega^2 - k^2}}$$

$$\text{where} \quad \arg(\omega^2 - k^2)^{\frac{1}{2}} = \begin{cases} 0 & \omega > k \\ -\frac{\pi}{2} & \omega < k \end{cases}$$

$$\text{and} \quad R = (\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi'))^{\frac{1}{2}} \quad ;$$

the addition theorem is

$$J_o(\omega R) = \sum_{n=0}^{\infty} (2 - \delta_{on})\,J_n(\omega\rho)\,J_n(\omega\rho')\cos[n(\phi - \phi')] \quad ,$$

$$\text{where} \quad \delta_{on} = \begin{matrix} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{matrix}$$

With these substitutions (18) gives

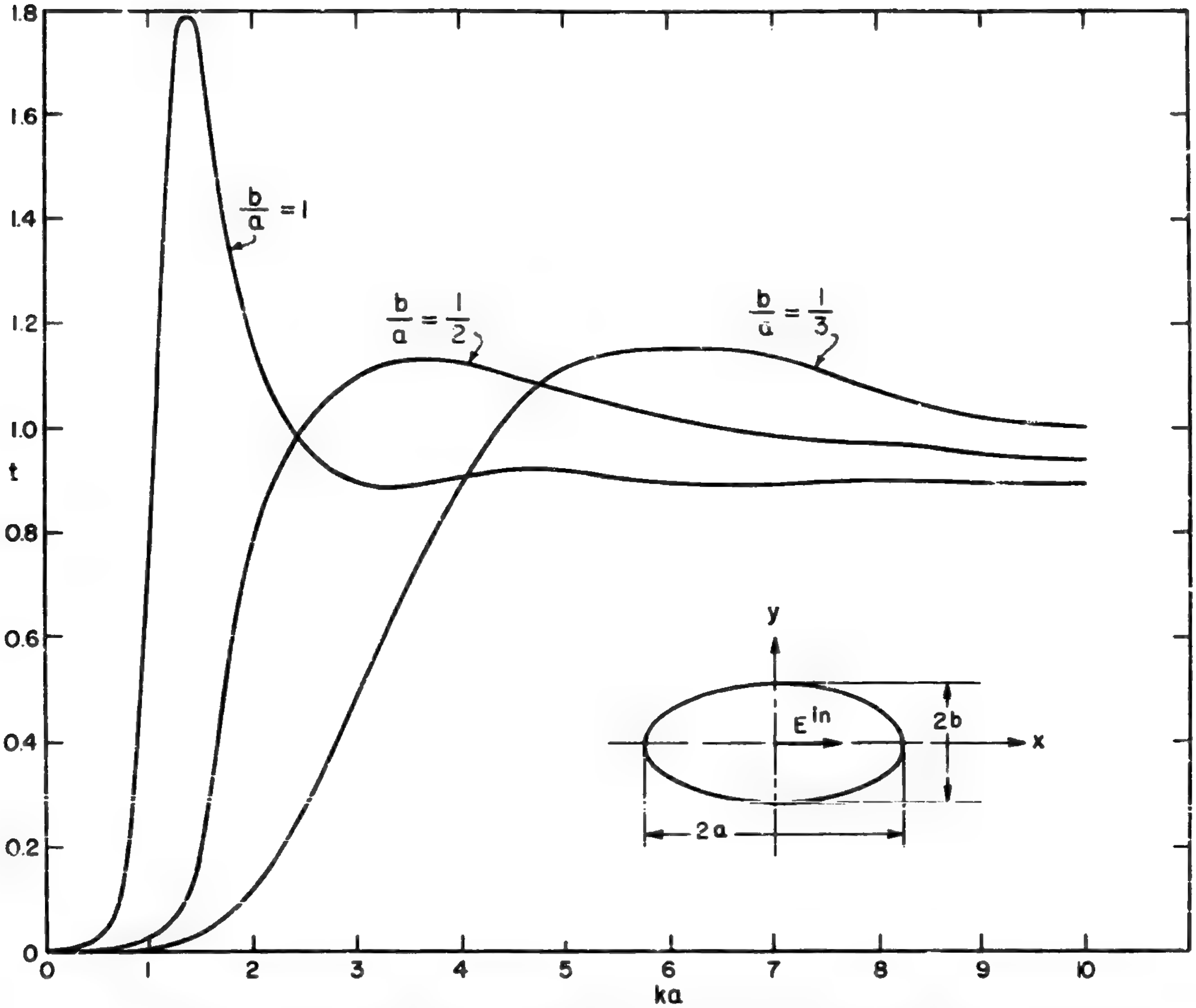

FIG. 4 TRANSMISSION COEFFICIENT OF ELLIPTIC APERTURE FOR NORMAL INCIDENCE OF PLANE ELECTROMAGNETIC WAVES, A VARIATIONAL APPROXIMATION BASED ON TANGENTIAL ELECTRIC APERTURE FIELD OF $\hat{x}\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)^{\frac{1}{2}}$

$$C^{aa}_{nm} = \frac{1}{4\pi}\int_0^a \frac{(\frac{\rho}{a})^{2n}}{\sqrt{1-(\frac{\rho}{a})^2}}\,\rho\,d\rho \int_0^a \frac{(\frac{\rho'}{a})^{2n}}{\sqrt{1-(\frac{\rho'}{a})^2}}\,\rho'\,d\rho' \int_0^{2\pi}\cos\phi\,d\phi \int_0^{2\pi}\cos\phi'\,d\phi'$$

$$[\cos(\phi-\phi') - \frac{1}{k^2}\,\frac{1}{\rho}\,\frac{\partial}{\partial\phi}\,\frac{1}{\rho'}\,\frac{\partial}{\partial\phi'}]\sum_{n=0}^{\infty}(2-\delta_{on})\cos[n(\phi-\phi')]$$

$$\cdot\int_0^{\infty}\frac{\omega\,d\omega}{\sqrt{\omega^2-k^2}}\;J_n(\omega\rho)\;J_n(\omega\rho')$$

The orthogonal properties of the trigonometric functions permit the integration with respect to ϕ and ϕ' to be carried out. The result is

$$C^{aa}_{nm} = \frac{\pi}{4}\int_0^{\infty}\frac{\omega\,d\omega}{\sqrt{\omega^2-k^2}}\int_0^a\frac{(\frac{\rho}{a})^{2n}}{\sqrt{1-(\frac{\rho}{a})^2}}\,\rho\,d\rho\int_0^a\frac{(\frac{\rho'}{a})^{2m}}{\sqrt{1-(\frac{\rho'}{a})^2}}\,\rho'\,d\rho'\,[J_o(\omega\rho)\,J_o(\omega\rho')$$

$$+ J_2(\omega\rho)\;J_2(\omega\rho') - \frac{2}{k^2}\,\frac{1}{\rho\,\rho'}\;J_1(\omega\rho)\;J_1(\omega\rho')] \quad .$$

By introducing the notation

$$I^a_{on} = \int_0^a \frac{(\frac{\rho}{a})^{2n}\,\rho\,d\rho}{\sqrt{1-(\frac{\rho}{a})^2}}\;J_o(\omega\rho) \quad ,$$

$$I^a_{2n} = \int_0^a \frac{(\frac{\rho}{a})^{2n}\,\rho\,d\rho}{\sqrt{1-(\frac{\rho}{a})^2}}\;J_2(\omega\rho)$$

the integral can be written as

$$C^{aa}_{nm} = \frac{\pi}{4}\int_0^{\infty}\frac{\omega\,d\omega}{\sqrt{\omega^2-k^2}}\left\{(1-\frac{\omega^2}{k^2})[I^a_{on}\,I^a_{om} + I^a_{2n}\,I^a_{2m}] + \frac{\omega^2}{k^2}[I^a_{on} - I^a_{2n}][I^a_{om} - I^a_{2m}]\right\}$$

$$I_{01}^{a} = \int_{o}^{a} \frac{\rho^{3} J_{o}(\omega a)\, d\rho}{a\sqrt{a^{2}-\rho^{2}}} = \int_{o}^{\pi/2} a^{2} \sin^{3}\theta\, J_{o}(\omega a \sin\theta)\, d\theta$$

$$= I_{00}^{a} - I_{00}^{b} = a^{2}\sqrt{\frac{\pi}{2}}\, [\frac{J_{1/2}(\omega a)}{(\omega a)^{1/2}}] - \frac{J_{3/2}(\omega a)}{(\omega a)^{3/2}}]$$

$$I_{20}^{a} = \int_{o}^{a} \frac{a\, \rho\, J_{2}(\omega a)\, d\rho}{\sqrt{a^{2}-\rho^{2}}} = \int_{o}^{\pi/2} a^{2} \sin\theta\, J_{2}(\omega a \sin\theta)\, d\theta$$

$$= a^{2} \int_{o}^{\pi/2} \sin\theta\, [\frac{2}{\omega a \sin\theta}\, J_{1}(\omega a \sin\theta) - J_{o}(\omega a \sin\theta)]\, d\theta$$

$$= a^{2}\sqrt{\frac{\pi}{2}}\, [\frac{2S_{1/2}(\omega a)}{(\omega a)^{3/2}} - \frac{J_{1/2}(\omega a)}{(\omega a)^{1/2}}]$$

$$I_{20}^{b} = \int_{o}^{a} \frac{\rho\sqrt{a^{2}-\rho^{2}}\, J_{2}(\omega\rho)\, d\rho}{a} = \int_{o}^{\pi/2} a^{2} \cos^{2}\theta \sin\theta\, J_{2}(\omega a \sin\theta)\, d\theta$$

$$= I_{20}^{a} - I_{21}^{a} = a^{2}\sqrt{\frac{\pi}{2}}\, [2\, \frac{S_{1/2}(\omega a)}{(\omega a)^{3/2}} - 3\, \frac{J_{3/2}(\omega a)}{(\omega a)^{3/2}}]$$

These forms for the I's are obtained by making use of Sonine's first finite integral[21]

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^{\nu}\Gamma(\nu+1)} \int_{o}^{\pi/2} J_{\mu}(z \sin\theta) \sin^{\mu+1}\theta \cos^{2\nu+1}\theta\, d\theta$$

$$\mathrm{Re}\ \mu, \nu > -1$$

and its alternate form

$$\int_{o}^{\pi/2} J_{n}(z \sin\theta) \sin^{1-\mu}\theta \cos^{2\nu+1}\theta\, d\theta = \frac{s_{\mu+\nu,\, \nu-\mu+1}(z)}{2^{\mu-1} z^{\nu+1}\Gamma(\mu)}\ , \quad \mathrm{Re}\ \nu > -1$$

where $s_{\mu\nu}(z)$ is a Lommel function. When the indices μ, ν are half integers these functions are defined by

$$s_{\mu,-\mu}(z) = \frac{\pi}{2} \sin \mu\pi [J_\mu(z) \int_0^z z^\mu J_{-\mu}(z)\, dz - J_{-\mu}(z) \int_0^z z^\mu J_\mu(z)\, dz]$$

and are related to the Struve function $S_\mu(z)$ in accordance with the equation[22]

$$s_{+\mu,-\mu}(z) = 2^\mu \Gamma(\tfrac{1}{2})\, \Gamma(\mu + \tfrac{1}{2})\; S_\mu(z)$$

When the integrated functions, I, are substituted into the expressions for the C's and these are combined according to equation (27), a certain amount of algebraic manipulation yields

$$P(a) = \frac{\pi^2 a^4}{4} \int_0^\infty \frac{\omega\, d\omega}{\sqrt{\omega^2 - k^2}} \left\{ \frac{J_{1/2}^2(\omega a)}{(\omega a)} - 4 \frac{J_{1/2}(\omega a)\, J_{3/2}(\omega a)}{(\omega a)^2} + 5 \frac{J_{3/2}^2(\omega a)}{(\omega a)^3} - \frac{\omega^2}{k^2} \frac{J_{3/2}^2(\omega a)}{(\omega a)^3} \right\}$$

$$Q(a) = \frac{\pi^2 a^4}{4} \int_0^\infty \frac{\omega\, d\omega}{\sqrt{\omega^2 - k^2}} \left\{ 2 \frac{J_{1/2}^2(\omega a)}{(\omega a)} - 6 \frac{J_{1/2}(\omega a)\, J_{3/2}(\omega a)}{(\omega a)^2} + 8 \frac{J_{3/2}^2(\omega a)}{(\omega a)^3} - 4 \frac{\omega^2}{k^2} \frac{J_{3/2}^2(\omega a)}{(\omega a)^3} \right\}$$

$$R(a) = \frac{\pi^2 a^4}{4} \int_0^\infty \frac{\omega\, d\omega}{\sqrt{\omega^2 - k^2}} \left\{ \frac{J_{1/2}^2(\omega a)}{(\omega a)} - 2 \frac{J_{1/2}(\omega a)\, J_{3/2}(\omega a)}{(\omega a)^2} + 5 \frac{J_{3/2}^2(\omega a)}{(\omega a)^3} - 4 \frac{\omega^2}{k^2} \frac{J_{3/2}^2(\omega a)}{(\omega a)^3} \right\}$$

It can be seen that each of these functions is a linear combination of the following integrals

$$I_1 = \int_0^\infty \frac{\omega\, d\omega}{\sqrt{\omega^2 - k^2}} \; \frac{J_{1/2}^2(\omega a)}{(\omega a)}$$

$$I_2 = \int_0^\infty \frac{\omega^3\, d\omega}{k^2 \sqrt{\omega^2 - k^2}} \; \frac{J_{3/2}^2(\omega a)}{(\omega a)^3}$$

$$I_3 = \int_0^\infty \frac{\omega\, d\omega}{\sqrt{\omega^2-k^2}} \left(1 - \frac{\omega^2}{k^2}\right) \frac{J_{3/2}^2(\omega a)}{(\omega a)^3}$$

$$I_4 = \int_0^\infty \frac{\omega\, d\omega}{\sqrt{\omega^2-k^2}} \frac{J_{1/2}(\omega a)\, J_{3/2}(\omega a)}{(\omega a)^2}$$

The remainder of this appendix will be concerned with evaluation of these integrals.

The integral I_1 can be rewritten as

$$I_1 = \frac{1}{a}\int_0^\infty \frac{J_{1/2}^2(\omega)\, d\omega}{\sqrt{\omega^2-a^2)}} = \frac{1}{a}\left\{\int_a^\infty \frac{J_{1/2}^2(\omega)\, d\omega}{\sqrt{\omega^2-a^2}} + i\int_0^a \frac{J_{1/2}^2(\omega)\, d\omega}{\sqrt{a^2-\omega^2}}\right\}$$

By making use of the integral representation[23]

$$J_\mu(z)\, J_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\, d\theta$$

$$\mathrm{Re}\,(\mu+\nu) > -1$$

The real part of I_1 becomes

$$\mathrm{Re}\, I_1 = \frac{2}{\pi a}\int_0^{\pi/2} d\theta \int_a^\infty \frac{J_1(2\omega\cos\theta)\, d\omega}{\sqrt{\omega^2-a^2}}$$

But with the infinite integral formula[24]

$$\int_k^\infty \frac{J_\nu(\omega a)\,(\omega^2-k^2)^\mu\, d\omega}{\omega^{\nu-1}} = \frac{2^\mu\,\Gamma(\mu+1)}{a^{\mu+1}k^{\nu-\mu-1}}\, J_{\nu-\mu-1}(ka) \qquad a \geqslant 0$$

$$\mathrm{Re}\left(\frac{\nu}{2}-\frac{1}{4}\right) > \mathrm{Re}\ \mu > -1$$

the integration with respect to ω can be carried out and the result is

$$\text{Re } I_1 = \frac{2}{\pi a} \int_0^{\pi/2} d\theta \sqrt{\frac{\pi}{2}} \quad \frac{J_{1/2}(2a \cos \theta)}{\sqrt{2a \cos \theta}}$$

We now make use of the fact that when ν is half of an odd integer, the function $J_\nu(z)$ has a finite representation in terms of algebraic and trigometric functions of z[25], i.e.

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin z .$$

Then $$\text{Re } I_1 = \frac{2}{\pi a} \int_0^{\pi/2} \frac{\sin (2a \cos \theta)\, d\theta}{2a \cos \theta} = \frac{2}{\pi a} \frac{1}{a} F_1(a) ,$$

where
$$F_1(a) = \int_0^{\pi/2} \frac{\sin (2a \cos \theta)\, d\theta}{2 \cos \theta}$$

It is evident that $F_1(0) = 0$ and that

$$F_1'(a) = \int_0^{\pi/2} \cos (2a \cos \theta)\, d\theta = \frac{\pi}{2} J_o(2a)$$

From these it follows that

$$F_1(a) = \int_0^a F'(a)\, da = \frac{\pi}{2} \int_0^a J_o(2a)\, da = \frac{\pi}{4} \int_0^{2a} J_o(t)\, dt .$$

Accordingly,

$$\text{Re } I_1 = \frac{1}{2a a} \int_0^{2a} J_o(t)\, dt .$$

The imaginary part of I_1 can be expressed by

$$\text{Im } I_1 = \frac{1}{a}\int_0^a \frac{J_{1/2}^2(\omega)\, d\omega}{\sqrt{a^2-\omega^2}} = \frac{2}{\pi a}\int_0^{\pi/2} d\theta \int_0^a \frac{J_1(2\omega\cos\theta)\, d\omega}{\sqrt{a^2-\omega^2}}$$

By making use of the second form of the Sonine's first finite integral[21], the integration with respect to ω can be carried out:

$$\text{Im } I_1 = \frac{2}{\pi a}\int_0^{\pi/2} d\theta \; \frac{s_{1/2,-1/2}(2a\cos\theta)}{\sqrt{2a\cos\theta}}$$

But[22]

$$s_{1/2,-1/2}(z) = z^{1/2}(1-\cos z)$$

Therefore the single integral can be rewritten as

$$\text{Im } I_1 = \frac{2}{\pi a}\int_0^{\pi/2} \frac{[1-\cos(2a\cos\theta)]\, d\theta}{2a\cos\theta} = \frac{2}{\pi a a} \; F_2(a)$$

where

$$F_2(a) = \int_0^{\pi/2} \frac{[1-\cos(2a\cos\theta)]\, d\theta}{2\cos\theta}$$

Again, with $F_2(0) = 0$ and with the integral representation for the Struve function[26]

$$S_\nu(z) = \frac{2(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi/2} \sin(z\cos\theta)\sin^{2\nu}\theta\, d\theta$$

$$\text{Re } \nu > -\frac{1}{2},$$

we have

$$F_2'(a) = \int_0^a \sin(2a\cos\theta)\, d\theta = \frac{\pi}{2} S_0(2a)$$

Consequently,

$$F_2(a) = \int_0^a F_2'(a)\,da = \frac{\pi}{2}\int_0^a S_o(2a)\,da = \frac{\pi}{4}\int_0^{2a} S_o(t)\,dt,$$

and

$$\text{Im } I_1 = \frac{1}{2aa}\int_0^{2a} S_o(t)\,dt \quad .$$

Therefore

$$I_1 = \frac{1}{2aa}\left[\int_0^{2a} J_o(t)\,dt + i\int_0^{2a} S_o(t)\,dt\right] .$$

The integral I_2 can be evaluated by decomposing it into two parts:

$$I_2 = \int_0^{\infty} \frac{\omega^3\,d\omega}{k^2\sqrt{\omega^2-k^2}}\;\frac{J_{3/2}^2(\omega a)}{\sqrt{\omega^2-k^2}} = \frac{1}{aa^2}\int_0^{\infty} \frac{d\omega\, J_{3/2}^2(\omega)}{\sqrt{\omega^2-a^2}}$$

$$= \frac{1}{aa^2}\int_0^{\infty} \frac{d\omega\, J_{3/2}(\omega)}{\sqrt{\omega^2-a^2}}\left[\frac{J_{1/2}(\omega)}{\omega} - J_{-1/2}(\omega)\right]$$

If we let

$$I_5 = \int_0^{\infty} \frac{d\omega\, J_{3/2}(\omega)\; J_{1/2}(\omega)}{\omega\sqrt{\omega^2-a^2}}$$

$$I_6 = \int_0^{\infty} \frac{d\omega\; J_{3/2}(\omega)\; J_{-1/2}(\omega)}{\sqrt{\omega^2-a^2}} \quad ,$$

then

$$I_2 = \frac{1}{aa^2}\left[I_5 - I_6\right]$$

The evaluation of I_5 proceeds in a manner similar to the one used for I_1. Thus

$$\text{Re } I_5 = \int_a^\infty \frac{J_{3/2}(\omega)\, J_{1/2}(\omega)\, d\omega}{\omega\sqrt{\omega^2 - a^2}} = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta\, d\theta \int_a^\infty \frac{J_2(2\omega\cos\theta)\, d\omega}{\omega\sqrt{\omega^2 - a^2}}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos\theta\, d\theta \sqrt{\frac{\pi}{2}}\ \frac{1}{\sqrt{2\cos\theta}\ a^{3/2}}\ J_{3/2}(2a\cos\theta)$$

$$= \frac{1}{\pi a^2}\int_0^{\pi/2} d\theta \left[\frac{\sin(2a\cos\theta)}{2a\cos\theta} - \cos(2a\cos\theta)\right]$$

$$= \frac{1}{\pi a^2}\left[\frac{1}{a}F_1(a) - \frac{\pi}{2}J_0(2a)\right] = \frac{1}{4a^2}\left[\frac{1}{a}\int_0^{2a} J_0(t)\, dt - 2J_0(2a)\right]$$

and,

$$\text{Im } I_5 = \int_0^a \frac{J_{3/2}(\omega)\, J_{1/2}(\omega) d\omega}{\omega\sqrt{a^2 - \omega^2}} = \frac{2}{\pi}\int_0^{\pi/2} \cos d\theta \int_0^a \frac{J_2(2\omega\cos\theta)\, d\omega}{\omega\sqrt{a^2 - \omega^2}}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{2a\sqrt{2a\cos\theta}}\quad s_{3/2,-3/2}(2a\cos\theta)$$

But

$$s_{3/2,-3/2}(z) = z^{1/2}\left[z - 2\sin z + \frac{2}{z}(1 - \cos z)\right]$$

Therefore,

$$\text{Im } I_5 = \frac{1}{\pi a^2}\int_0^{\pi/2} d\theta \left[a\cos\theta - \sin(2a\cos\theta) + \frac{(1 - \cos(2a\cos\theta))}{2a\cos\theta}\right]$$

$$= \frac{1}{\pi a^2}\left[a - \frac{\pi}{2} S_o(2a) + \frac{1}{a} F_2(a) \right]$$

$$= \frac{1}{4a^2}\left[\frac{4}{\pi} a - 2 S_o(2a) + \frac{1}{a}\int_o^{2a} S_o(t)\, dt \right]$$

Similarly

$$\mathrm{Re}\, I_6 = \int_a^{\infty} \frac{d\omega\, J_{3/2}(\omega) J_{-1/2}(\omega)}{\sqrt{\omega^2 - a^2}} = \frac{2}{\pi}\int_o^{\pi/2} \cos 2\theta\, d\theta \int_a^{\infty} \frac{J_1(2\omega\cos\theta)\, d\omega}{\sqrt{\omega^2 - a^2}}$$

$$= \frac{2}{\pi}\int_o^{\pi/2} \cos 2\theta\, d\theta \sqrt{\frac{\pi}{2}}\; \frac{1}{\sqrt{2a\cos\theta}}\; J_{1/2}(2a\cos\theta)$$

$$= \frac{2}{\pi}\int_o^{\pi/2} \cos 2\theta\, d\theta\; \frac{\sin(2a\cos\theta)}{2a\cos\theta} = \frac{2}{\pi a} F_3(a)$$

where

$$F_3(a) = \int_o^{\pi/2} \cos 2\theta\; \frac{\sin(2a\cos\theta)}{2\cos\theta}\; d\theta \; .$$

Since $F_3(0) = 0$ and

$$F_3'(a) = \int_o^{\pi/2} \cos 2\theta \cos(2a\cos\theta)\, d\theta = -\frac{\pi}{2} J_2(2a) \; ,$$

therefore

$$F_3(a) = \int_o^{a} F_3'(a)\, da = -\frac{\pi}{4}\int_o^{2a} J_2(t)\, dt = -\frac{\pi}{4}\int_o^{2a} J_o(t)\, dt + \frac{\pi}{2} J_1(2a)$$

and

$$\mathrm{Re}\, I_6 = \frac{-1}{2a}\int_o^{2a} J_o(t)\, dt + \frac{\pi}{2} J_1(2a) \; .$$

On the other hand

$$\operatorname{Im} I_6 = \int_0^a \frac{d\omega\, J_{3/2}(\omega)\, J_{-1/2}(\omega)}{\sqrt{a^2-\omega^2}} = \frac{2}{\pi}\int_0^{\pi/2} \cos 2\theta\, d\theta \int_0^a \frac{J_1(2\omega\cos\theta)\, d\theta}{\sqrt{a^2-\omega^2}}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos 2\theta \;\frac{[1-\cos(2a\cos\theta)]\, d\theta}{2a\cos\theta}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} (1-2\sin^2\theta)\,\frac{[1-\cos(2a\cos\theta)]\, d\theta}{2a\cos\theta} = \frac{2}{\pi a}[F_2(a)-2F_4(a)]$$

where $F_4(a) = \int_0^{\pi/2} \sin^2\theta \;\dfrac{[1-\cos(2a\cos\theta)]\, d\theta}{2\cos\theta}$.

Again $F_4(0) = 0$

$$F_4'(a) = \int_0^{\pi/2} \sin^2\theta\, \sin(2a\cos\theta)\, d\theta = \frac{\pi}{4a} S_1(2a) \;;$$

therefore

$$F_4(a) = \frac{\pi}{4}\int_0^{2a} \frac{S_1(t)\, dt}{t} \;.$$

If the recurrence formula for the Struve function is used[27]

$$z S'_\nu(z) = -\nu S_\nu(z) + z S_{\nu-1}(z)$$

we have $F_4(a) = \dfrac{\pi}{4}\left[\int_0^{2a} S_0(t)\, dt - S_1(2a)\right]$.

Consequently $\operatorname{Im} I_6 = \dfrac{1}{2a}\left[2S_1(2a) - \int_0^{2a} S_0(t)\, dt\right]$

Combining I_5 and I_6, we obtain

$$I_2 = \frac{1}{ah^2}\ [I_5 - I_6]$$

$$= \frac{1}{a a^2}\left\{[\frac{1}{a}(\frac{1}{2}+\frac{1}{4a^2})\int_0^{2a} J_o(t)\,dt - \frac{1}{2}J_1(2a) - \frac{1}{2a^2}\ J_o(2a)]\right.$$

$$\left. + i\,[\frac{1}{\pi a} + \frac{1}{a}(\frac{1}{2}+\frac{1}{4a^2})\int_0^{2a} S_o(t)\,dt - \frac{1}{a}S_1(2a) - \frac{1}{2a^2}\ S_o(2a)]\right\} .$$

The next integral to be considered is

$$I_3 = \int_0^{\infty}\frac{\omega\,d\omega}{\sqrt{\omega^2-k^2}}\ (1-\frac{\omega^2}{k^2})\ \frac{J_{3/2}^2(\omega a)}{(\omega a)^3}$$

$$= \frac{1}{a a^2}\int_0^{\infty} d\,(\frac{1}{\omega})\ \sqrt{\omega^2-k^2}\ \ J_{3/2}^2\ (\omega a) .$$

With an integration by parts

$$I_3 = \frac{-1}{a a^2}\ [\ 2a\int_0^{\infty}\frac{\sqrt{\omega^2-k^2}}{\omega}\ \ J_{3/2}(\omega a)\ J'_{3/2}(\omega a)\ d\omega + \int_0^{\infty}\frac{J_{3/2}^2(\omega a)}{\sqrt{\omega^2-k^2}}\ ]$$

But

$$J'_{3/2}(z) = -\frac{3}{2}\,\frac{J_{3/2}(z)}{z} + J_{1/2}(z) ,$$

therefore

$$I_3 = \frac{1}{a a^2}\ [\ 3\int_0^{\infty}\frac{\sqrt{\omega^2-k^2}\ \ J_{3/2}^2(\omega a)\ d\omega}{\omega^2} - 2a\int_0^{\infty}\frac{\sqrt{\omega^2-k^2}\ \ J_{3/2}(\omega a)\ J_{1/2}(\omega a)\ d\omega}{\omega}$$

$$-\int_0^{\infty} \frac{J_{3/2}^2(\omega a)\, d\omega}{\sqrt{\omega^2-k^2}} \Big] = -3I_3 - \frac{2}{a a^2} I_7 - I_2$$

and

$$I_3 = -\frac{1}{2aa^2} I_7 - \frac{1}{4} I_2$$

where

$$I_7 = \int_0^{\infty} \frac{a\sqrt{\omega^2-k^2}\, J_{3/2}(\omega a)\, J_{1/2}(\omega a)\, d\omega}{\omega} = \int_0^{\infty} \frac{\sqrt{\omega^2-a^2}\, J_{3/2}(\omega)\, J_{1/2}(\omega)\, d\omega}{\omega}$$

But it is easy to show that

$$I_7 = \frac{1}{4a}\int_0^{2a} J_o(t)\, dt + i\left[\frac{-a}{\pi} + \frac{1}{4a}\int_0^{2a} S_o(t)\, dt\right] .$$

Therefore when I_2 and I_7 are combined, we find that

$$I_3 = \frac{-1}{aa^2}\Bigg\{\left[\frac{1}{4a}\left(1+\frac{1}{4a^2}\right)\int_0^{2a} J_o(t)\, dt - \frac{1}{8a^2} J_o(2a) - \frac{1}{4a} J_1(2a)\right]$$

$$+ i\left[\frac{-a}{2\pi} + \frac{1}{4\pi a} + \frac{1}{4a}\left(1+\frac{1}{4a^2}\right)\int_0^{2a} S_o(t)\, dt - \frac{1}{8a^2} S_o(2a) - \frac{1}{4a} S_1(2a)\right]\Bigg\}$$

The last integral, I_4, is directly related to I_5 as follows

$$I_4 = \int_0^{\infty} \frac{\omega\, d\omega}{\sqrt{\omega^2-k^2}} \frac{J_{3/2}(\omega a)\, J_{1/2}(\omega a)}{(\omega a)^2} = \frac{1}{a}\int_0^{\infty} \frac{J_{1/2}(\omega)\, J_{3/2}(\omega)}{\omega\sqrt{\omega^2-a^2}} = \frac{1}{a} I_5 .$$

Therefore

$$I_4 = \frac{1}{4aa^2}\left\{\left[\frac{1}{a}\int_0^{2a} J_o(t)\, dt - 2J_o(2a)\right] + i\left[\frac{4}{\pi}a + \frac{1}{a}\int_0^{2a} S_o(t)\, dt - 2S_o(2a)\right]\right\}$$

Finally, by substituting the integrals I_1, I_2, I_3, I_4 into the function $P(a)$ $Q(a)$ and $R(a)$, we obtain

$$P(a) = \frac{\pi^2 a^3}{4a^2}\Bigg\{\Big[\Big(2 - \frac{11}{8}\frac{1}{a^2}\Big) J_o(2a) - \frac{11}{4}\frac{1}{a} J_1(2a) + \Big(\frac{a}{2} - \frac{1}{4}\frac{1}{a} + \frac{11}{16}\frac{1}{a^3}\Big)$$

$$\cdot \int_o^{2a} J_o(t)\,dt\Big] + i\Big[\frac{-3a}{2\pi} + \frac{11}{4\pi}\frac{1}{a} + \Big(2 - \frac{11}{8}\frac{1}{a^2}\Big) S_o(2a) - \frac{11}{4}\frac{1}{a} S_1(2a)$$

$$+ \Big(\frac{a}{2} - \frac{1}{4}\frac{1}{a} + \frac{11}{16}\frac{1}{a^3}\Big) \int_o^{2a} S_o(t)\,dt\Big]\Bigg\}$$

$$Q(a) = \frac{\pi^2 a^3}{4a^2}\Bigg\{\Big[\Big(3 - \frac{1}{a^2}\Big) J_o(2a) - \frac{2}{a} J_1(2a) + \Big(a - \frac{3}{2}\frac{1}{a} + \frac{1}{2a^3}\Big)\int_o^{2a} J_o(t)\,dt\Big]$$

$$+ i\Big[\frac{-2a}{\pi} + \frac{2}{\pi a} + \Big(3 - \frac{1}{a^2}\Big) S_o(2a) - \frac{2}{a} S_1(2a)$$

$$+ \Big(a - \frac{3}{2}\frac{1}{a} + \frac{1}{2a^3}\Big) \int_o^{2a} S_o(t)\,dt\Big]\Bigg\}$$

$$R(a) = \frac{\pi^2 a^3}{4a^2}\Bigg\{\Big[\Big(1 + \frac{1}{8}\frac{1}{a^2}\Big) J_o(2a) + \frac{1}{4a} J_1(2a) + \Big(\frac{a}{2} - \frac{5}{4}\frac{1}{a} - \frac{1}{16}\frac{1}{a^3}\Big)\int_o^{2a} J_o(t)\,dt\Big]$$

$$+ i\Big[\frac{a}{2\pi} - \frac{1}{4\pi a} + \Big(1 + \frac{1}{8}\frac{1}{a^2}\Big) S_o(2a) + \frac{1}{4a} S_1(2a)$$

$$+ \Big(\frac{a}{2} - \frac{5}{4}\frac{1}{a} - \frac{1}{16}\frac{1}{a^3}\Big) \int_o^{2a} S_o(t)\,dt\Big]\Bigg\}.$$

Appendix B

In this appendix, a form of the transmission coefficient for the elliptical apertures which is appropriate to small values of ka will be derived.

When the expansions (36) of the functions $F(x)$ and $G(x)$ are substituted into (35), and the integrations with respect to θ are carried out term by term, we find that

$$I = \frac{b^2 \pi}{4k^2 a} \Big\{ [R_0(e) + R_2(e)(ka)^2 + R_4(e)(ka)^4 + \cdots$$

$$+ i\,[I_0(e) + I_2(e)(ka)^2 + I_4(e)(ka)^4 + \cdots] \Big\}$$

where $R_0(e) = \frac{-4}{3e^2}\,[K(e) - K(e)]$

$$R_2(e) = \frac{8}{15} K(e) - \frac{4}{15e^2}\,[K(e) - E(e)]$$

$$R_4(e) = \frac{-4}{35} E(e) + \frac{3}{35e^2}\,[E(e) - E(e)]$$

$$I_o = \frac{+8}{27}$$

$$I_2(e) = \frac{-8}{675}\,[3 - e^2]$$

$$I_4(e) = \frac{+8}{55125}\,[16 - 10\,e^2 + 3\,e^4]$$

and e = eccentricity $= \sqrt{1 - (\frac{b}{a})^2}$

$K(e)$ = complete elliptic integral of the first kind[28]

$$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - e^2 \sin^2\theta}}$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 e^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 e^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 e^6 + \cdots\right]$$

$E(e)$ = complete elliptic integral of the second kind

$$= \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\theta}\; d\theta$$

$$= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{e^6}{5} - \cdots\right]$$

$$\bar{K}(e) = \int_0^{\pi/2} \frac{d\theta}{(1 - e^2\sin^2\theta)^{3/2}} = -\frac{1}{2}\frac{1}{e^2}\frac{d}{d(\frac{1}{e})}[e K(e)] = -e^2 \frac{d}{de}\left(\frac{K}{e}\right)$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 3e^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 5e^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 7e^6 + \cdots\right]$$

$$\bar{E}(e) = \int_0^{\pi/2} (1 - e^2\sin^2\theta)^{3/2} d\theta = -e^3\left[3\int_1^e \frac{E(e)}{e^4}\, de + \frac{2}{3}\right]$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 \frac{3}{(-1)(1)} e^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{3}{1\cdot 3} e^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{3}{3\cdot 5} e^6\right.$$

$$\left. + \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2 \frac{3}{5\cdot 7} e^8 + \left(\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\right)^2 \frac{3}{7\cdot 9} e^{10} + \cdots\right]$$

Substitution of I into (32) gives

$$t = \frac{8a^4}{9}\,\frac{a}{b}\,\frac{+[I_0 + I_2\alpha^2 + I_4\alpha^4 + \cdots]}{R_0^2 + 2R_0R_2\alpha^2 + (R_2^2 + 2R_0R_4)\alpha^4 + (I_0^2 + 2R_2R_4)\alpha^6 + \cdots}$$

where $\alpha = ka$

or

$$t = \frac{64a^4}{27\pi^2}\left(\frac{a}{b}\right)[A_0(e) + A_2(e)\alpha^2 + A_4(e)\alpha^4 + \cdots]$$

where $A_0(e) = \frac{+3\pi^2}{8} \frac{I_0}{R_0^2(e)}$

$$A_2(e) = \frac{+3\pi^2}{8R_0^2(e)} [I_2(e) - \frac{2R_2(e)}{R_0(e)} I_0]$$

$$A_4(e) = \frac{+3\pi^2}{8R_0^2(e)} [I_4(e) - \frac{2R_2(e)}{R_0(e)} I_2 + \frac{3R_2^2(e) - 2R_0(e)R_4(e)}{R_0^2(e)} I_0]$$

Acknowledgement

The work described in this report was carried out under the general supervision of Prof. R. W. P. King whose encouragement and help made its completion possible. The author would like also to acknowledge his debt of gratitude to Dr. H. Levine who initially suggested the topic for study and has generously given his time for discussions and advices; to Dr. R. D. Kodis who has made many valuable suggestions for the preparation of the manuscript.

In addition thanks are due to Miss Lesley Gerould and Miss Mary Tynan who carried out many of the rather tedious calculations.

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